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# Stationary axion/dilaton solutions and supersymmetry

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## Abstract

We present a new set of supersymmetric stationary solutions of pure  $N = 4, d = 4$  supergravity (and, hence, of low-energy effective string theory) that generalize (and include) the Israel–Wilson–Perjés solutions of Einstein–Maxwell theory. All solutions have  $1/4$  of the supersymmetries unbroken and some have  $1/2$ . The full solution is determined by two arbitrary complex harmonic functions  $\mathcal{H}_{1,2}$  which transform as a doublet under  $SL(2, \mathbb{R})$   $S$  duality and  $N$  complex constants  $k^{(n)}$  that transform as an  $SO(N)$  vector. This set of solutions is, then, manifestly duality invariant. When the harmonic functions are chosen to have only one pole, all the general resulting point-like objects have supersymmetric rotating asymptotically Taub–NUT metrics with  $1/2$  or  $1/4$  of the supersymmetries unbroken. The static, asymptotically flat metrics describe supersymmetric extreme black holes. Only those breaking  $3/4$  of the supersymmetries have regular horizons. The stationary asymptotically flat metrics do not describe black holes when the angular momentum does not vanish, even in the case in which  $3/4$  of the supersymmetries are broken.

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## 1. Introduction

The theory of pure  $N = 4, d = 4$  supergravity without vector multiplets presents an interesting and relatively simple model to study various bosonic solutions with unbroken supersymmetry. It has a richer structure than pure  $N = 2, d = 4$  supergravity and, in

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particular, this theory allows for configurations with either  $1/2$  or  $1/4$  of the  $N = 4$  supersymmetries unbroken, while in  $N = 2, d = 4$  supergravity only  $1/2$  of the  $N = 2$  supersymmetries can be unbroken [1]. This is related to the fact that the central charge of the  $N = 2$  theory is replaced in the  $N = 4$  theory by a central charge *matrix*. In the appropriate basis, the  $N = 4$  supersymmetry algebra gives rise to two Bogomol'nyi bounds that can be saturated independently [2]. When only one is saturated, only  $1/4$  of the supersymmetries are unbroken. When both are simultaneously saturated  $1/2$  of the  $N = 4$  supersymmetries are unbroken.

Static axion/dilaton black holes with  $1/4$  of the supersymmetries unbroken were first found in Refs. [3–5]. The most general family of solutions, presented in Ref. [5], describes an arbitrary number of extreme black holes with (almost) arbitrary electric and magnetic charges in the six  $U(1)$  gauge groups and non-trivial complex moduli, i.e. non-trivial dilaton and axion fields which are combined in a single complex scalar field. The constraint that the charges have to satisfy is a Bogomol'nyi identity and it is related to the existence of unbroken supersymmetry [6,1,2,7]. The solutions of this family generically have  $1/4$  of the supersymmetries unbroken, but some (in fact a whole subfamily) have  $1/2$  of the supersymmetries unbroken.

Much work has also been devoted to the so-called axion/dilaton gravity theory which is a truncation of  $N = 4, d = 4$  supergravity with only one vector field [8–15]. However, as we will extensively discuss, the presence of only one vector field is insufficient to generate all the interesting metrics. In particular, in the supersymmetric limit, with only one vector field one cannot get those with  $1/4$  of the supersymmetries unbroken.

Another interesting feature of pure  $N = 4, d = 4$  supergravity is that it is the simplest model that exhibits both  $S$  and  $T$  dualities. The group  $SL(2, \mathbb{R})$  (quantum-mechanically broken to  $SL(2, \mathbb{Z})$ ) acts on the vector fields by interchanging electric and magnetic fields and acts on the complex scalar by fractional-linear transformations that, in particular, include the inversion of the complex scalar. In the old supergravity days these transformations were not physically understood. They just were simply there. However, in the framework of string theory the complex scalar has a physical meaning because it contains the string coupling constant (the dilaton) and it was conjectured in Refs. [16,17] that this symmetry could be a non-perturbative symmetry relating the strong and weak-coupling regimes of string theory:  $S$  duality.

The  $T$  duality group is  $SO(6)$  (again, quantum-mechanically broken to  $SO(6, \mathbb{Z})$ ) and rotates amongst them the six Abelian vector fields of  $N = 4, d = 4$  supergravity. Its physical meaning can only be found in string theory.

The family of solutions presented in Ref. [5] is invariant under both dualities and, therefore, it contains all the solutions that can be generated by using them. Taking into account the number of charges that can be assigned arbitrarily to each black hole and the fact that a Bogomol'nyi bound has to be saturated in order to have unbroken supersymmetry it is reasonably to expect that these are the most general static black-hole solutions of this theory with unbroken supersymmetry.

An interesting aspect of the static black-hole solutions with unbroken supersymmetry in pure  $N = 4$  supergravity is the intriguing relation that seems to hold between the

number of unbroken supersymmetries and the finiteness of the area of the horizon. All those with only  $1/4$  of the supersymmetries unbroken have finite area, while those with  $1/2$  do not have a regular horizon.<sup>4</sup>

But the static ones are not the only solutions with unbroken supersymmetry of  $N = 4, d = 4$  supergravity. Some stationary solutions are also known [8], but all of them have  $1/2$  of the supersymmetries unbroken and none of them represents a black hole with regular horizon. It is clear, though, that stationary solutions with  $1/4$  of the supersymmetries unbroken must exist. In particular, it must be possible to embed the stationary solutions of pure  $N = 2, d = 4$  supergravity into  $N = 4, d = 4$  supergravity. These were found in Ref. [19] and turned out to be the Israel–Wilson–Perjés (IWP) [21] family of metrics. It is a general feature that solutions of  $N = 2, d = 4$  supergravity with  $1/2$  of the supersymmetries unbroken have  $1/4$  of them unbroken when embedded in  $N = 4, d = 4$ . The best-known example [22,3] is the extreme Reissner–Nordström (RN) black hole and the entire Majumdar–Papapetrou (MP) [23] family of metrics describing many extreme RN black holes in equilibrium [24]. Therefore, one would expect to find the IWP metrics as solutions of  $N = 4, d = 4$  supergravity with  $1/4$  of the supersymmetries unbroken, and we will present here the explicit embedding.

The main goal of this paper is to find the most general class of supersymmetric stationary solutions of  $N = 4, d = 4$  supergravity, which should include the so-far unknown stationary solutions with  $1/4$  of the supersymmetries unbroken (among them the IWP metrics), those with  $1/2$  presented in Ref. [8] as a particular case and all the static solutions of Ref. [5] in the static limit. We will present and study this general class of solutions that we will call SWIP because they generalize the IWP solutions.

We can point out already at this stage that the correspondence between finiteness of the black-hole area and  $1/4$  of the supersymmetries unbroken will not hold for the stationary solutions, because the IWP metrics are part of them and they are known to be singular except in the static limit [24]. The best-known example of this fact is the Kerr–Newman metric, which reaches the extreme limit  $m - |q| = J$  much before it reaches the supersymmetric limit  $m = |q|$ . In this limit it has a naked singularity and it is an IWP solution. Similar results have been found in the context of the low-energy heterotic string effective action [25] which is equivalent to  $N = 4, d = 4$  supergravity coupled to 22 vector multiplets.

A further reason to study the simple model of  $N = 4, d = 4$  supergravity is the fact that any solution of this theory can be embedded into  $N = 8, d = 4$  supergravity, whose solutions are interesting from the point of view of  $U$  duality [26]. By looking for the most general supersymmetric solutions with  $1/4$  of unbroken supersymmetry in  $N = 4$

<sup>4</sup> This relation no longer exist when matter is added to the pure supergravity theory or in other supergravities ( $N = 8$ ). An example is provided by the extreme  $a = 1/\sqrt{3}$  dilaton black hole, which is a solution of the low-energy heterotic string theory compactified in a six-torus ( $N = 4$  supergravity coupled to 22 vector multiplets), which has a singular horizon and  $1/4$  of the  $N = 4$  supersymmetries unbroken. This black hole is also a solution of the  $N = 8$  supergravity theory with only  $1/8$  of the supersymmetries (i.e. the same as in  $N = 4$ ) unbroken [18].

theory we may make some progress in the problem of finding the most general solutions with  $1/4$  and  $1/8$  of unbroken supersymmetry in the  $N = 8$  theory [18].

On the other hand, a reduced version of  $N = 4$  supergravity with only two vector fields, offers a nice example of  $N = 2$  supergravity interacting with one vector multiplet. Our  $N = 4$  solutions will supply us with solutions of the  $N = 2$  theory with one vector multiplet with  $1/2$  of unbroken supersymmetry. Supersymmetric solutions in  $N = 2$  supergravity coupled to vector multiplets and hyper-multiplets are poorly understood. Only solutions of pure  $N = 2$  supergravity [19] as well as the ones reduced from  $N = 4$  supergravity are known. In more general cases, when vector multiplets are included, only magnetic black-hole solutions are known [27]. The analysis of all supersymmetric stationary solutions of  $N = 4$  supergravity performed in this paper will allow us to derive some lessons for the study of generic electric and magnetic solutions of  $N = 2$  theory.

It will be particularly useful to reinterpret the results for the  $N = 4$  theory in terms of the Kähler geometry of the  $N = 2$  theory.<sup>5</sup> We will find that for all of our new stationary solutions, the metric is described as in [27], in terms of the Kähler potential  $K(X, \bar{X})$ . However, in addition to that, it depends on a chiral  $U(1)$  connection  $A_\mu$  which breaks hypersurface orthogonality and makes the solutions stationary. The appearance of this special geometry object has not been realized before. However, since both the Kähler potential as well as the  $U(1)$  connection are invariant under symplectic transformations, it is not surprising that both of these functions show up in the canonical metric of the most general duality-invariant family of solutions. This suggests how to find the most general stationary supersymmetric solutions of  $N = 2, d = 4$  supergravity coupled to an arbitrary number of vector multiplets. We will comment more on this in Section 2.5.

Finally, we would like to remark that point-like objects (black holes among them) are just one type of the many objects described by the metrics that we are about to present: by taking two complex harmonic functions that depend on only two or one spatial coordinate one gets strings or domain walls, although we will not study them here.

This paper is organized as follows. In Section 2 we describe the SWIP solutions of pure  $N = 4, d = 4$  supergravity. In Section 2.1 we present the action and our conventions, in Section 2.2 we present the solutions, in Section 2.3 we describe their relation with previously known solutions and their behavior under duality transformations is described in Section 2.4. In Section 2.5 we discuss these solutions from the point of view of stationary supersymmetric solutions of  $N = 2$  supergravity coupled to vector multiplets. In Section 3 we study the most general single point-like object described by a SWIP solution. In Section 3.1 we study those with NUT charge and no angular momentum (extreme axion/dilaton Taub-NUT solutions) and in Section 3.2 we study asymptotically flat (i.e. zero NUT charge) rotating solutions showing that there are

<sup>5</sup> Static solutions of  $N = 4$  theory have been identified as solutions of  $N = 2$  supergravity interacting with the vector multiplet before [27].

no supersymmetric rotating black holes (that is, with regular horizon) with  $1/4$  or more supersymmetries unbroken in  $N = 4, d = 4$  supergravity. In Section 4 we prove explicitly the supersymmetry of the point-like SWIP metrics and find their Killing spinors. Section 5 contains our conclusions.

## 2. General axion/dilaton IWP solution

### 2.1. $N = 4, d = 4$ supergravity

Our conventions are those of Refs. [3,8] and are summarized in an appendix of the first reference, the only difference being that world indices are underlined instead of carrying a hat. Our theory contains a complex scalar  $\lambda = a + ie^{-2\phi}$  that parametrizes an  $SL(2, \mathbb{R})$  coset. When we consider this theory as part of a low-energy effective string theory,  $a$  is the axion field (the dual of the usual two-form axion field  $B_{\mu\nu}$ ) and  $\phi$  is the dilaton field. It also contains the Einstein metric  $g_{\mu\nu}$  and an arbitrary number  $N$  of  $U(1)$  vector fields  $A_\mu^{(n)}$ ,  $n = 1, 2, \dots, N$ .

The action is

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left\{ -R + 2(\partial\phi)^2 + \frac{1}{2}e^{4\phi}(\partial a)^2 - \sum_{n=1}^N [e^{-2\phi} F^{(n)} F^{(n)} + ia F^{(n)} \star F^{(n)}] \right\}, \quad (1)$$

When the total number of vector fields is six, this action is identical to the bosonic part of the action of  $N = 4, d = 4$  supergravity. When the total number of vector fields is greater than six, this action does not correspond to any supergravity action: additional vector multiplets of  $N = 4, d = 4$  supergravity would have additional scalars. We prefer to leave the number of vector fields arbitrary for the sake of generality.

Sometimes it is convenient to use alternative ways of writing this action:

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left\{ -R + \frac{1}{2} \frac{\partial_\mu \lambda \partial^\mu \bar{\lambda}}{(\text{Im } \lambda)^2} - i \sum_{n=1}^N F^{(n)} \star \tilde{F}^{(n)} \right\} \quad (2)$$

$$= \frac{1}{16\pi} \int d^4x \sqrt{-g} \left\{ -R + \frac{1}{2} \frac{\partial_\mu \lambda \partial^\mu \bar{\lambda}}{(\text{Im } \lambda)^2} + 2 \text{Re} \left( i\lambda \sum_{n=1}^N F^{(n)+} F^{(n)+} \right) \right\}, \quad (3)$$

where we have defined the  $SL(2, \mathbb{R})$ -duals<sup>6</sup> to the fields  $F_{\mu\nu}^{(n)} = \partial_\mu A_\nu^{(n)} - \partial_\nu A_\mu^{(n)}$

$$\tilde{F}^{(n)} = e^{-2\phi} \star F^{(n)} - ia F^{(n)}. \quad (4)$$

<sup>6</sup> The space-time duals are  $\star F^{(n)\mu\nu} = \frac{1}{2\sqrt{-g}} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ , with  $\epsilon^{0123} = \epsilon_{0123} = +i$ .

The advantage of using  $\tilde{F}^{(n)}$  is that the equations of motion for the vector fields can be written in this way

$$d^* \tilde{F}^{(n)} = 0, \quad (5)$$

and imply the local existence of  $N$  real vector potentials  $\tilde{A}^{(n)}$  such that

$$\tilde{F}_{\mu\nu}^{(n)} = i (\partial_\mu \tilde{A}_\nu^{(n)} - \partial_\nu \tilde{A}_\mu^{(n)}) . \quad (6)$$

The analogous equations  $F_{\mu\nu}^{(n)} = \partial_\mu A_\nu^{(n)} - \partial_\nu A_\mu^{(n)}$  are a consequence of the Bianchi identities

$$d^* F^{(n)} = 0 \quad (7)$$

(or, obviously, the definition of  $F_{\mu\nu}^{(n)}$ ).

If the time-like components  $A_t^{(n)}$  play the role of electrostatic potentials, then the  $\tilde{A}_t^{(n)}$ 's will play the role of magnetostatic potentials. A virtue of this formalism is that the duality rotations can be written in terms of the vector fields  $A^{(n)}$  and  $\tilde{A}^{(n)}$  instead of the field strengths:<sup>7</sup>

$$\begin{aligned} A^{(n)'}(x) &= \delta A^{(n)}(x) - \gamma \tilde{A}^{(n)}(x), \\ \tilde{A}^{(n)'}(x) &= -\beta A^{(n)}(x) + \alpha \tilde{A}^{(n)}(x), \end{aligned} \quad (8)$$

where  $\alpha, \beta, \gamma$  and  $\delta$  are the elements of an  $SL(2, \mathbb{R})$  matrix

$$R = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \quad (9)$$

Note that, since the  $\tilde{A}^{(n)}$ 's are not independent fields, the consistency of Eqs. (8) implies the usual transformation law of  $\lambda$ ,

$$\lambda'(x) = \frac{\alpha \lambda(x) + \beta}{\gamma \lambda(x) + \delta}. \quad (10)$$

With no dilaton nor axion ( $\lambda = i$ ), our theory coincides with the Einstein–Maxwell theory. In this case  $\tilde{F} = *F$  and the consistency of Eqs. (8) would imply that  $R$  is an  $SO(2)$  matrix, the duality group being just  $U(1)$ .

## 2.2. The SWIP solutions

Now let us describe the solutions. All the functions entering in the different fields can ultimately be expressed in terms of two completely arbitrary complex harmonic functions  $\mathcal{H}_1(x)$  and  $\mathcal{H}_2(x)$ ,

$$\partial_{\underline{t}} \partial_{\underline{t}} \mathcal{H}_1 = \partial_{\underline{t}} \partial_{\underline{t}} \mathcal{H}_2 = 0, \quad (11)$$

<sup>7</sup> Of course, this is only valid on-shell, where the  $\tilde{A}^{(n)}$ 's exist, but  $SL(2, \mathbb{R})$  is only a symmetry of the equations of motion anyway.

and a set of complex constants  $k^{(n)}$  that satisfy the constraints

$$\sum_{n=1}^N (k^{(n)})^2 = 0, \quad \sum_{n=1}^N |k^{(n)}|^2 = \frac{1}{2}, \quad (12)$$

in the general case.<sup>8</sup> This means that in general we must have at least two non-trivial vector fields.

The harmonic functions enter through the following two combinations into the metric:<sup>9</sup>

$$e^{-2U} = 2 \operatorname{Im} (\mathcal{H}_1 \bar{\mathcal{H}}_2), \quad (13)$$

$$\partial_{[\underline{i}} \omega_{\underline{j}]} = \epsilon_{ijk} \operatorname{Re} (\mathcal{H}_1 \partial_{\underline{k}} \bar{\mathcal{H}}_2 - \bar{\mathcal{H}}_2 \partial_{\underline{k}} \mathcal{H}_1). \quad (14)$$

The fields themselves are

$$ds^2 = e^{2U} (dt^2 + \omega_{\underline{i}} dx^{\underline{i}})^2 - e^{-2U} d\mathbf{x}^2, \quad (15)$$

$$\lambda = \frac{\mathcal{H}_1}{\mathcal{H}_2}, \quad (16)$$

$$A_t^{(n)} = 2e^{2U} \operatorname{Re} (k^{(n)} \mathcal{H}_2), \quad (17)$$

$$\tilde{A}_t^{(n)} = -2e^{2U} \operatorname{Re} (k^{(n)} \mathcal{H}_1). \quad (18)$$

Given the time components  $A_t^{(n)}$ ,  $\tilde{A}_t^{(n)}$ , all the components of the “true” vector fields  $A_\mu^{(n)}$  are completely determined.

### 2.3. Relation with previously known solutions

First we observe that these solutions reduce to the usual IWP metrics [21] when

$$\mathcal{H}_1 = i\mathcal{H}_2 = \frac{1}{\sqrt{2}} V^{-1} \quad (19)$$

(in the notation of Ref. [19]), where  $V^{-1}$  is a complex harmonic function, but now embedded in  $N = 4$  supergravity. All these metrics admit Killing spinors when embedded in  $N = 2$  supergravity [19] and generically have 1/2 of the supersymmetries of  $N = 2$  supergravity unbroken [20]. In fact, apart from  $pp$ -waves, all supersymmetric solutions of  $N = 2, d = 4$  supergravity have IWP metrics [19].

However, the only black-hole-type solutions (i.e. describing point-like objects with regular horizons covering the singularities) in this class are the MP solutions Ref. [23] ( $V = \bar{V}$  with the right asymptotics) that describe an arbitrary number of extreme RN black holes in static equilibrium [24]. Any amount of angular momentum added to the MP solutions produces naked singularities. One can also add NUT charge (general IWP metrics have angular momentum and NUT charge), but then the spaces are not

<sup>8</sup> If  $\mathcal{H}_1$  or  $\mathcal{H}_2$  are constant, then only the second constraint is necessary.

<sup>9</sup> Here  $\epsilon_{123} = +1$ .



asymptotically flat and do not admit a black-hole interpretation. There are no supersymmetric rotating black holes in pure  $N = 2, d = 4$  supergravity and, in consequence, in the IWP subclass of the SWIP solutions of  $N = 4, d = 4$  supergravity there will be no supersymmetric rotating black holes either. We will discuss later how many  $N = 4, d = 4$  unbroken supersymmetries they have.

If we now take

$$\mathcal{H}_1 = \frac{i}{\sqrt{2}}\lambda, \quad \mathcal{H}_2 = \frac{i}{\sqrt{2}}, \quad (20)$$

we recover the electric-type axion–dilaton IWP solutions of Ref. [8]. All these solutions have  $1/2$  of all  $N = 4$  supersymmetries unbroken. Again, all solutions with angular momentum in this class have naked singularities. Observe that this class does not contain any  $N = 2$  IWP metric and is intrinsically  $N = 4$ .

Finally, if we impose the constraint

$$\partial_{[\underline{i}}\omega_{j]} = 0 \Rightarrow \text{Re}(\mathcal{H}_1\partial_{\underline{k}}\bar{\mathcal{H}}_2 - \bar{\mathcal{H}}_2\partial_{\underline{k}}\mathcal{H}_1) = 0, \quad (21)$$

which gives static metrics, we recover the solutions of Ref. [5]. The existence of a constraint on the harmonic functions in those solutions is implicit in the form in which the two harmonic functions were given. However, when those solutions were found by a lengthy and not transparent process of covariantization with respect to  $SL(2, \mathbb{R})$  duality rotations (first discussed in this context in Ref. [28]) it was almost impossible to see that the relations between the two complex harmonic functions could be expressed as just the effect of imposing that the metrics are static, as opposite to stationary.

The black-hole solutions in this class have either  $1/2$  or  $1/4$  of the supersymmetries unbroken [3,29]. The complexity of the constraints between the complex harmonic functions made a proof of the supersymmetry of the general solution very difficult to obtain. This class trivially contains all the static  $N = 2$  IWP metrics, and also the usual extreme dilaton black holes [30–32] from which they were obtained by generalization [4] and  $SL(2, \mathbb{R})$  covariantization [5].

Finally, we should mention the relation with solutions of the low-energy heterotic string theory compactified on a six-torus [33]. From the supergravity point of view, this theory is nothing but  $N = 4, d = 4$  supergravity coupled to 22 vector supermultiplets. Our solution is related to truncations in which all 22 matter vector fields (and scalars) vanish. This truncation corresponds to the case in which the six vector fields that come from the ten-dimensional axion are identified (up to a convention-dependent sign) with the six vector fields that come from the ten-dimensional metric and the remaining 16 vector fields vanish. The most general static solution of the low-energy heterotic string effective action compactified on a six-torus and *given in terms of independent harmonic functions* was recently found in Ref. [34] and rediscovered in Ref. [35]. The truncation typically reduces the number of independent harmonic functions by a half. Then, some of the static solutions in the SWIP class, with only two real independent harmonic functions correspond to some of the solutions in Ref. [34] with two independent harmonic

functions. It is difficult to make more accurate comparisons between these two families of metrics because they are given in two very different settings.

Another stationary solution of the low-energy heterotic string effective action depending on just one real harmonic function  $F^{-1}$  was recently constructed in Ref. [36]. This solution breaks 1/2 of the supersymmetries and, when the matter vector fields are set to zero, they correspond to one of the  $N = 4$  solutions in Ref. [8].

We will say more about this and other solutions in Section 3 where we will discuss the relation between other known solutions and the SWIP metrics in the single point-like object (spherically or axially symmetric) case.

## 2.4. Duality properties

Since our general SWIP solution is a straightforward generalization of the solutions in Ref. [5], it shares many of their properties, in particular those concerning its behavior under  $SL(2, \mathbb{R})$  transformations: the whole family transforms into itself under  $SL(2, \mathbb{R})$ . Individual solutions in this family are simply interchanged by  $SL(2, \mathbb{R})$  transformations, and, therefore, the effect of the transformations can be expressed by substituting  $\mathcal{H}_{1,2}$  and the  $k^{(n)}$ s by its *primed* counterparts:  $\mathcal{H}'_{1,2}$  and  $k^{(n)'}.$

So, what is the action of  $SL(2, \mathbb{R})$  on them?  $\mathcal{H}_{1,2}$  transform as a doublet (that is, linearly) under  $SL(2, \mathbb{R})$  while the  $k^{(n)}$ 's are invariant. To be more precise, observe first that  $\mathcal{H}_1, \mathcal{H}_2$  and the  $k^{(n)}$ 's are defined up to a complex phase: if we multiply  $\mathcal{H}_1$  and  $\mathcal{H}_2$  by the same complex phase and the constants  $k^{(n)}$  by the opposite phase, the solution is invariant. An  $SL(2, \mathbb{R})$  rotation as we have defined it transforms  $\mathcal{H}_{1,2}$  as a doublet up to a complex phase and scales the  $k^{(n)}$ 's by the opposite phase but we can always absorb that phase as we just explained.

Incidentally, the solution is also  $SO(N)$  covariant: the constants  $k^{(n)}$  (and, hence, the  $N$   $U(1)$  vector fields) transform as an  $SO(N)$  vector.<sup>10</sup>

Therefore, this solution has all the duality symmetries of  $N = 4, d = 4$  supergravity “built in” and nothing more general can be generated by duality rotations. In a sense they are the “true” generalization of the IWP solutions which also have built in the duality symmetries of  $N = 2, d = 4$  supergravity: a  $U(1)$  electric–magnetic duality rotation of the single vector field of  $N = 2, d = 4$  supergravity (the only duality symmetry of this theory) corresponds to multiplying the complex harmonic function  $V^{-1}$  by a complex phase, giving another IWP solution with the same metric, etc.

On the other hand, all these metrics have a time-like isometry and one can perform further  $T$  duality [37] transformations in the time direction. We will not attempt to study here the result of these transformations.

<sup>10</sup>  $SO(6, \mathbb{Z})$  is the duality group of the pure  $N = 4$  truncation of heterotic string theory compactified on a six-torus.

### 2.5. Relation with general $N = 2$ solutions

In Ref. [27], whose conventions we follow in this section, some static extremal magnetic black-hole solutions of  $N = 2, d = 4$  supergravity coupled to an arbitrary number of vector multiplets were constructed in the case in which the ratios of the complex  $X^A$  are real. The metric turns out to depend exclusively on the Kähler potential  $K(Z, \bar{Z})$ , where  $Z^A = X^A/X^0$ :

$$ds^2 = e^{2U} dt^2 - e^{-2U} d\mathbf{x}^2, \quad e^{2U} = e^{K(Z, \bar{Z}) - K_\infty}. \quad (22)$$

It was then realized that the solutions presented in Ref. [5] could also be described in terms of the Kähler potential associated to pure  $N = 4, d = 4$  supergravity with only two vector fields interpreted as  $N = 2, d = 4$  supergravity coupled to a single vector multiplet. For this theory, the prepotential is  $F(X) = 2X^0 X^1$ . Taking  $X^0$  and  $X^1$  to be two suitably normalized complex harmonic functions

$$X^0 = i\mathcal{H}_2, \quad X^1 = \mathcal{H}_1, \quad (23)$$

one immediately arrives at the Kähler potential

$$\begin{aligned} e^{-K(X, \bar{X})} &= \bar{X}^A N_{A\Sigma} X^\Sigma \\ &= 2 \operatorname{Im}(\mathcal{H}_1 \bar{\mathcal{H}}_2) \\ &= e^{-2U}. \end{aligned} \quad (24)$$

Therefore the Kähler potential provides the factor  $e^{2U}$  in the metric. With the complex harmonic functions  $\mathcal{H}_{1,2}$  constrained as in Ref. [5], this is all there is to it. However, if the harmonic functions  $\mathcal{H}_{1,2}$  are not constrained, there is another geometrical object which is a symplectic invariant and which in this case does not vanish: the chiral connection

$$\begin{aligned} A_\mu &= \frac{i}{2} N_{A\Sigma} \left[ \bar{X}^A \partial_\mu X^\Sigma - (\partial_\mu \bar{X}^A) X^\Sigma \right] \\ &= -2 \operatorname{Re} (\mathcal{H}_1 \partial_\mu \bar{\mathcal{H}}_2 - \bar{\mathcal{H}}_2 \partial_\mu \mathcal{H}_1) \\ &= -\delta_\mu{}^k \epsilon_{ijk} \partial_{[i} \omega_{j]}. \end{aligned} \quad (25)$$

Thus, the chiral connection naturally gives us the three-vector  $\omega_i$  which codifies the information about angular momentum, NUT charge, etc. Imposing the condition that the metric is static can be simply expressed as the vanishing of the chiral connection:

$$A_\mu = 0. \quad (26)$$

It is surprising to some extent and highly non-trivial that this constraint does not constrain the values of the electric and magnetic charges and the masses and that they are essentially arbitrary (as long as the Bogomol'nyi bound is saturated). This constraint seems to have a purely (space-time) geometrical content.

We see that all the elements appearing in the general metric of the SWIP solution have a special geometrical meaning. Hence, it is only natural to try to extend this scheme to more general  $N = 2, d = 4$  theories with an arbitrary number of vector multiplets and arbitrary prepotential  $F(X)$ . We conjecture that the metric of most general supersymmetric stationary solution of  $N = 2, d = 4$  supergravity coupled to vector multiplets can be written in the form

$$ds^2 = e^{K(X, \bar{X})} (dt^2 + \omega_i dx^i)^2 - e^{-K(X, \bar{X})} d\mathbf{x}^2, \quad (27)$$

where  $K(X, \bar{X})$  is the Kähler potential associated to the corresponding holomorphic prepotential  $F(X)$  and  $\omega_i$  is determined by the chiral connection  $A_\mu$  as before. A natural step that would generalize the results obtained in Ref. [27] for static purely magnetic solution would be to obtain static dyonic solutions by using the constraint of the vanishing of the chiral connection.

### 3. Point-like SWIP solutions

The most general choice of complex harmonic functions for a point-like object is

$$\mathcal{H}_1 = \chi_0 + \frac{\chi_1}{r_1}, \quad \mathcal{H}_2 = \psi_0 + \frac{\psi_1}{r_2}, \quad (28)$$

where  $\chi_0, \chi_1, \psi_0, \psi_1$  are arbitrary complex constants and  $r_{1,2}^2 = (\mathbf{x} - \mathbf{x}_{1,2}) \cdot (\mathbf{x} - \mathbf{x}_{1,2})$  where the constants  $\mathbf{x}_{1,2}$  are arbitrary and complex. Up to shifts in the coordinate  $z$ , the most general possibility compatible with having a single point-like object is  $r_1^2 = r_2^2 = x^2 + y^2 + (z - i\alpha)^2$ .

We have then at our disposal  $9 + 2n$  real integration constants, including the  $k^{(n)}$ 's. After imposing the normalization of the metric at infinity and the three real constraints that the  $k^{(n)}$ 's have to satisfy in order to have a solution, it seems that we are left with  $5 + 2n$  independent integration constants. However, if one multiplies both  $\mathcal{H}_{1,2}$  by the same complex phase, the solution remains invariant, and so we only have  $4 + 2n$  meaningful independent integration constants at our disposal with  $n \geq 2$  in the generic case.

On the other hand, the maximum number of independent physical parameters that we can have seems to be <sup>11</sup>  $5 + 2n$ : the mass  $m$ , the NUT charge  $l$ , the angular momentum  $J$ , the complex moduli  $\lambda_0$  (which is the value of the complex scalar at infinity) and  $2n$  electric and magnetic charges  $(q^{(n)}, p^{(n)})$ . However, when the field configurations have unbroken supersymmetry, there is at least one constraint between them: the Bogomol'nyi identity. Therefore we only expect  $4 + 2n$  independent physical parameters in the supersymmetric case. We are going to show that all these solutions satisfy the Bogomol'nyi identity and, since the number of independent physical parameters matches the number of integration constants, we expect them to be the most general axisymmetric solutions of our theory with (at least) 1/4 of the supersymmetries unbroken.

<sup>11</sup> Although some sort of "no-hair" theorem probably holds for this theory, none has yet been proven.

Studying the asymptotic behavior of the different fields (see later) we find that the integration constants are related to the physical parameters of the solution<sup>12</sup> by

$$\begin{aligned}\chi_0 &= \frac{1}{\sqrt{2}} e^{\phi_0} \lambda_0 e^{i\beta}, & \chi_1 &= \frac{1}{\sqrt{2}} e^{\phi_0} e^{i\beta} (\lambda_0 \mathcal{M} + \bar{\lambda}_0 \mathcal{T}), \\ \psi_0 &= \frac{1}{\sqrt{2}} e^{\phi_0} e^{i\beta}, & \psi_1 &= \frac{1}{\sqrt{2}} e^{\phi_0} e^{i\beta} (\mathcal{M} + \mathcal{T}), \\ k^{(n)} &= -e^{-i\beta} \frac{\mathcal{M} \Gamma^{(n)} + \overline{\mathcal{T}} \Gamma^{(n)}}{|\mathcal{M}|^2 - |\mathcal{T}|^2}, & \alpha &= J/m,\end{aligned}\quad (29)$$

where  $J$  is the angular momentum and  $\beta$  is an arbitrary real number which does not play any physical role (but transforms under  $S$  duality according to our prescription).

Observe that these identifications have been made under the assumption  $m \neq 0$ . There are also massless solutions in this class.

The functions  $\mathcal{H}_1, \mathcal{H}_2, e^{-2U}$  and  $\omega_i$  take the form

$$\mathcal{H}_1 = \frac{1}{\sqrt{2}} e^{\phi_0} \lambda_0 e^{i\beta} \left( \lambda_0 + \frac{\lambda_0 \mathcal{M} + \bar{\lambda}_0 \mathcal{T}}{r} \right), \quad (30)$$

$$\mathcal{H}_2 = \frac{1}{\sqrt{2}} e^{\phi_0} e^{i\beta} \left( 1 + \frac{\mathcal{M} + \mathcal{T}}{r} \right), \quad (31)$$

$$e^{-2U} = 1 + 2 \operatorname{Re} \left( \frac{\mathcal{M}}{r} \right) + \frac{(|\mathcal{M}|^2 - |\mathcal{T}|^2)}{|r|^2}, \quad (32)$$

$$\partial_{[i} \omega_{j]} = \epsilon_{ijk} \operatorname{Im} \left[ \mathcal{M} \partial_{\underline{k}} \frac{1}{r} + (|\mathcal{M}|^2 - |\mathcal{T}|^2) \frac{1}{r} \partial_{\underline{k}} \frac{1}{\bar{r}} \right]. \quad (33)$$

That the above identifications between physical parameters and integration constants are correct becomes evident when we switch from Cartesian to oblate spheroidal coordinates  $(\rho, \theta, \varphi)$ ,

$$\begin{cases} x \pm iy = \sqrt{\rho^2 + \alpha^2} \sin \theta e^{\pm i\varphi}, \\ z = \rho \cos \theta, \\ dx^2 = (\rho^2 + \alpha^2 \cos^2 \theta)(\rho^2 + \alpha^2)^{-1} d\rho^2 + (\rho^2 + \alpha^2 \cos^2 \theta) d\theta^2 \\ \quad + (\rho^2 + \alpha^2) \sin^2 \theta d\varphi^2, \end{cases} \quad (34)$$

in which only the component  $\omega_\varphi$  does not vanish and takes the form<sup>13</sup>

$$\begin{aligned}\omega_\varphi &= 2 \cos \theta l + \alpha \sin^2 \theta (e^{-2U} - 1) \\ &= 2 \cos \theta l + \alpha \frac{\sin^2 \theta}{\rho^2 + \alpha^2 \cos^2 \theta} [2m\rho + 2l\alpha \cos \theta + (|\mathcal{M}|^2 - |\mathcal{T}|^2)],\end{aligned}\quad (35)$$

<sup>12</sup> Again we use the same definitions for electric and magnetic charges as in Ref. [5]. The only difference is that the mass  $m$  has to be substituted by the complex combination  $\mathcal{M} = m + il$ , where  $l$  is the NUT charge, in the definition of the complex scalar charge  $\mathcal{T} = -2 \sum_n \bar{\Gamma}^{(n)2} / \mathcal{M}$ .

<sup>13</sup> Observe that  $\rho$  can take positive or negative values since its sign is not determined by the coordinate transformation. We have used  $r = \rho + i\alpha \cos \theta$ .

while  $e^{-2U}$  takes the form

$$e^{-2U} = \frac{\rho^2 + \alpha^2 \cos^2 \theta + 2m\rho + 2l\alpha \cos \theta + (|\mathcal{M}|^2 - |\mathcal{T}|^2)}{\rho^2 + \alpha^2 \cos^2 \theta}. \quad (36)$$

The full general metric can be written in the standard way after a shift in the radial coordinate  $\hat{\rho} = \rho + m + |\mathcal{T}|$ ,

$$ds^2 = \frac{\Delta - \alpha^2 \sin^2 \theta}{\Sigma} (dt - \omega d\varphi)^2 - \Sigma \left( \frac{d\hat{\rho}^2}{\Delta} + d\theta^2 + \frac{\Delta \sin^2 \theta d\varphi^2}{\Delta - \alpha^2 \sin^2 \theta} \right), \quad (37)$$

where

$$\omega = -\omega_\varphi = \frac{2}{\Delta - \alpha^2 \sin^2 \theta} \{ l\Delta \cos \theta + \alpha \sin^2 \theta [m(\hat{\rho} - (m + |\mathcal{T}|)) + \frac{1}{2}(|\mathcal{M}|^2 - |\mathcal{T}|^2)] \}, \quad (38)$$

$$\Delta = \hat{\rho}[\hat{\rho} - 2(m + |\mathcal{T}|)] + \alpha^2 + (m + |\mathcal{T}|)^2, \quad (39)$$

$$\Sigma = \hat{\rho}(\hat{\rho} - 2|\mathcal{T}|) + (\alpha \cos \theta + l)^2. \quad (40)$$

Before comparing this metric with other rotating Taub-NUT solutions in the literature, we make the following observation: by construction, the following relation between the physical parameters is always obeyed in this class of solutions:

$$|\mathcal{M}|^2 + |\mathcal{T}|^2 - 4 \sum_n |\Gamma^{(n)}|^2 = 0. \quad (41)$$

This is the usual expression of the Bogomol'nyi identity in pure  $N = 4$  supergravity [4], and it is valid for solutions with 1/2 or 1/4 of the supersymmetries unbroken. Multiplying by  $|\mathcal{M}|^2$  and using the expression of  $\mathcal{T}$  in terms of  $\mathcal{M}$  and the  $\Gamma^{(n)}$ 's we observe that we can always rewrite it as follows:

$$(|\mathcal{M}|^2 - |\mathcal{Z}_1|^2)(|\mathcal{M}|^2 - |\mathcal{Z}_2|^2) = 0, \quad (42)$$

where  $|\mathcal{Z}_1|$  and  $|\mathcal{Z}_2|$  can be identified with the two different skew eigenvalues of the central charge matrix [1,2]. The above identity indicated that one of the two possible Bogomol'nyi bounds,

$$|\mathcal{M}|^2 \geq |\mathcal{Z}_{1,2}|^2, \quad (43)$$

is always saturated and, therefore, 1/4 of the supersymmetries of pure  $N = 4$  supergravity are always unbroken.

For only two vector fields, the central charge eigenvalues are linear in electric and magnetic charges:

$$\mathcal{Z}_1 = \sqrt{2}(\Gamma^{(1)} + i\Gamma^{(2)}), \quad (44)$$

$$\mathcal{Z}_2 = \sqrt{2}(\Gamma^{(1)} - i\Gamma^{(2)}), \quad (45)$$

but, in general, we have the non-linear expression<sup>14</sup>

$$\frac{1}{2}|\mathcal{Z}_{1,2}|^2 = \sum_n |\Gamma^{(n)}|^2 \pm \left[ \left( \sum_n |\Gamma^{(n)}|^2 \right)^2 - \left| \sum_n \Gamma^{(n)2} \right|^2 \right]^{1/2}. \quad (46)$$

When both central-charge eigenvalues are equal,  $|\mathcal{Z}_1| = |\mathcal{Z}_2|$ , and, therefore, 1/2 of the supersymmetries are unbroken, it is easy to prove that

$$|\mathcal{M}|^2 = |\mathcal{T}|^2 = |\mathcal{Z}_{1,2}|^2 = 2 \left| \sum_n \Gamma^{(n)2} \right|. \quad (47)$$

Now we are ready to compare the metric (37)–(40) with other general rotating asymptotically Taub-NUT metrics of  $N = 4, d = 4$  supergravity solutions. The most general metric of this kind was given in Eqs. (31)–(35) of Ref. [10] and has the same general form (37) but now with

$$\omega = \frac{2}{\Delta - \alpha^2 \sin^2 \theta} \left\{ l \Delta \cos \theta + \alpha \sin^2 \theta [m(r - r_-) + l(l - l_-)] \right\}, \quad (48)$$

$$\Delta = (r - r_-)(r - 2m) + \alpha^2 + (l - l_-)^2, \quad (49)$$

$$\Sigma = r(r - r_-) + (\alpha \cos \theta + l)^2 - l_-^2, \quad (50)$$

where the constants  $r_-$ ,  $l_-$  are related to the electric charge  $\mathcal{Q}$ , the mass and Taub-NUT charge by

$$r_- = \frac{m|\mathcal{Q}|^2}{|\mathcal{M}|^2}, \quad l_- = \frac{l|\mathcal{Q}|^2}{2|\mathcal{M}|^2}. \quad (51)$$

First, note that this metric is not supersymmetric in general. It becomes supersymmetric when  $|\mathcal{M}| = \sqrt{2}|\mathcal{Q}|$ . It is now very easy to check that in this limit and shifting the radial coordinate  $r = \hat{r} + m - |\mathcal{M}|$  one recovers the metric (37)–(40) with  $|\mathcal{M}|^2 - |\mathcal{T}|^2 = 0$ . Therefore, the supersymmetric limit of this metric (37)–(50), is a particular case of the SWIP metric with 1/2 of the supersymmetries unbroken.

If we compare with the axion/dilaton IWP solutions presented in Ref. [8], we observe that 1/2 of the supersymmetries were also always unbroken. This meant again that the terms proportional to the difference  $|\mathcal{M}|^2 - |\mathcal{T}|^2$  in the metric were always absent. The presence of these terms in the solutions that we are going to study, which implies the breaking of an additional 1/4 of the supersymmetries, proves crucial for the existence of regular horizons in the static cases. It is also easy to see that the complex scalar  $\lambda$  is also regular on the horizon when only 1/4 of the supersymmetries are unbroken: if  $|\mathcal{M}| \neq |\mathcal{T}|$ , then  $\mathcal{M} \neq \mathcal{T}$  and the constant in the denominator of

$$\lambda = \frac{\lambda_0 r + \lambda_0 \mathcal{M} + \overline{\lambda_0} \mathcal{T}}{r + \mathcal{M} + \mathcal{T}} \quad (52)$$

<sup>14</sup> We stress that only when there are six or less vector fields  $\mathcal{Z}_{1,2}$  have an interpretation in terms of pure  $N = 4, d = 4$  supergravity.

never vanishes. Then, on the would-be horizon (which we expect to be generically placed at  $r = 0$ ),  $\lambda$  takes the finite value

$$\lambda_{\text{horizon}} = \frac{\lambda_0 \mathcal{M} + \bar{\lambda}_0 \mathcal{T}}{\mathcal{M} + \mathcal{T}}. \quad (53)$$

As we will see, in the rotating case, the additional 1/4 of broken supersymmetries does not help in getting regular horizons, though.

From all this discussion we conclude that the most general solution in the SWIP class describing a point-like object is an asymptotically Taub-NUT metric with angular momentum. In general we expect to have at least 1/4 of the supersymmetries unbroken and a general proof will be given later in Section 4 with an explicit calculation of the Killing spinors. This is the main difference with previously known solutions. (We will compare with rotating asymptotically flat solutions in Section 3.2.)

Instead of studying the most general case, we will study separately two important particular cases: the non-rotating axion/dilaton Taub-NUT solution ( $\alpha = 0, l \neq 0$ ) and the rotating asymptotically flat solution ( $\alpha \neq 0, l \neq 0$ ).

### 3.1. Extreme axion/dilaton Taub-NUT solution

When the angular momentum  $J = m\alpha$  vanishes (but the mass  $m$  does not vanish), using the coordinate  $\rho$  is more adequate, and the solution takes the form

$$ds^2 = \left(1 + \frac{2m}{\rho} + \frac{|\mathcal{M}|^2 - |\mathcal{T}|^2}{\rho^2}\right)^{-1} (dt + 2l \cos \theta d\varphi)^2 \\ - \left(1 + \frac{2m}{\rho} + \frac{|\mathcal{M}|^2 - |\mathcal{T}|^2}{\rho^2}\right) (d\rho^2 + \rho^2 d\Omega^2), \quad (54)$$

$$\lambda = \frac{\lambda_0 r + \lambda_0 \mathcal{M} + \bar{\lambda}_0 \mathcal{T}}{r + \mathcal{M} + \mathcal{T}}. \quad (55)$$

The expressions for the  $A_t^{(n)}$ 's and  $\tilde{A}_t^{(n)}$ 's are quite complicated and can be readily obtained from the general solution.

This is the most general extreme axion/dilaton Taub-NUT solution [38] obtained so far. If we compare the metric with the metric of the extreme Taub-NUT solution in Eq. (14) of Ref. [8] we can immediately see that the difference is the additional  $(|\mathcal{M}|^2 - |\mathcal{T}|^2)/\rho^2$  term in  $e^{-2U}$ . This term vanishes when both central charge eigenvalues are equal and 1/2 of the supersymmetries are unbroken. This is always the case when there is only one vector field, as in Refs. [8,10], etc. The effect of this additional term is dramatic: when it is absent, the coordinate singularity at  $\rho = 0$  is also a curvature singularity. One can easily see that the area of the two-spheres of constant  $t$  and  $\rho$  is  $4\pi\rho(\rho + 2m)$  and it goes to zero when  $\rho$  goes to zero. If the additional term that breaks an additional 1/4 of the supersymmetries is present, in the limit  $\rho = 0$  one finds a finite area:

$$A = 4\pi(|\mathcal{M}|^2 - |\mathcal{T}|^2). \quad (56)$$



In fact, it is easy to rewrite the area formula in terms of the central-charge eigenvalues

$$A = 4\pi ||Z_1|^2 - |Z_2|^2|, \quad (57)$$

making evident that, when there is 1/2 of unbroken supersymmetry ( $|Z_1| = |Z_2|$ ), and only then, the area vanishes.<sup>15</sup>

On the other hand, with only one vector field one cannot set to zero the scalar charge, because it is equal to  $\mathcal{M}$ . Only with at least two vector fields one can set  $T = 0$  and recover solutions of  $N = 2$  supergravity (all in the IWP class), in particular the charged NUT metric of Ref. [39].

When  $l \neq 0$  this metric does not admit a black-hole interpretation since it has additional naked singularities along the axes  $\theta = 0, \pi$ , which can be removed using Misner's procedure [40] at the expense of changing the asymptotics and introducing closed time-like curves.

When  $l = 0$  this is the area of the extreme black-hole horizon [3,5]. When only 1/4 of the supersymmetries are unbroken and  $|\mathcal{M}| \neq |T|$ , this area is finite, the scalars are regular on this surface and  $\rho = 0$  does not correspond to a point. An infinite throat of finite section appears in this limit.

Finally, we observe that the area formula can be written in terms of the conserved charges<sup>16</sup>  $\tilde{q}^{(n)}, \tilde{p}^{(n)}$  as follows:

$$A = 8\pi \sqrt{(\tilde{q} \cdot \tilde{q})(\tilde{p} \cdot \tilde{p}) - (\tilde{q} \cdot \tilde{p})^2}. \quad (60)$$

In this formula, the independence of the area of the horizon of the string coupling constant and the moduli is evident, but the manifest duality invariance seems to be lost.<sup>17</sup> It is, though, not too complicated to rewrite yet again the area formula in a manifestly duality-invariant and moduli-independent fashion:

$$A = 8\pi \sqrt{\det \left[ \begin{pmatrix} \tilde{p}^t \\ \tilde{q}^t \end{pmatrix} (\tilde{p} \tilde{q}) \right]}, \quad (61)$$

where the action of the duality group on the charge vector  $\begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix}$  is given by

<sup>15</sup> We stress again that this is a property of *pure*  $N = 4$  supergravity that disappears when there is matter.

<sup>16</sup> The conserved charges are defined by

$$*F_{tr}^{(n)} \sim i \frac{\tilde{p}^{(n)}}{r^2}, \quad (e^{-2\phi} F - ia^* F)_{tr}^{(n)} \sim \frac{\tilde{q}^{(n)}}{r^2}, \quad (58)$$

and with them we build the charge vectors

$$\tilde{q} = \begin{pmatrix} \tilde{q}^{(1)} \\ \vdots \\ \tilde{q}^{(N)} \end{pmatrix}, \quad \tilde{p} = \begin{pmatrix} \tilde{p}^{(1)} \\ \vdots \\ \tilde{p}^{(N)} \end{pmatrix}. \quad (59)$$

<sup>17</sup> The invariance under duality of Eq. (57) is manifest because duality transformations only permute the absolute value of the central-charge eigenvalues.

$$\begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix}' = R \otimes S \begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix}, \quad (62)$$

where  $R$  is an  $SO(6)$  rotation matrix and  $S$  is an  $SL(2, \mathbb{R})$  (unimodular) matrix.

### 3.2. Rotating asymptotically flat solution

When the NUT charge vanishes ( $l = 0$ ) in the general solution we get

$$ds^2 = \frac{\Delta - \alpha^2 \sin^2 \theta}{\Sigma} (dt - \omega d\varphi)^2 - \Sigma \left( \frac{d\hat{\rho}^2}{\Delta} + d\theta^2 + \frac{\Delta \sin^2 \theta d\varphi^2}{\Delta - \alpha^2 \sin^2 \theta} \right), \quad (63)$$

where

$$\omega = -\omega_\varphi = \frac{2\alpha \sin^2 \theta}{\Delta - \alpha^2 \sin^2 \theta} \left\{ m[\hat{\rho} - (m + |\mathcal{T}|)] + \frac{1}{2}(m^2 - |\mathcal{T}|^2) \right\}, \quad (64)$$

$$\Delta = \hat{\rho}[\hat{\rho} - 2(m + |\mathcal{T}|)] + \alpha^2 + (m + |\mathcal{T}|)^2, \quad (65)$$

$$\Sigma = \hat{\rho}(\hat{\rho} - 2|\mathcal{T}|) + \alpha^2 \cos^2 \theta. \quad (66)$$

Again, the expressions for the potentials  $A_i^{(n)}$  and  $\tilde{A}_i^{(n)}$  are complicated and we refer the reader to the general expression.

When  $\alpha = 0$ , we recover the same general class of static black holes as in the previous section for  $l = 0$ . When  $m = |\mathcal{T}|$  this metric is essentially the one in Eq. (31) of Ref. [8] which has naked singularities. In this limit we also recover the metric of the solution in Eqs. (3.11)–(3.16) of Ref. [36] (after going to the string frame). Other rotating solutions of the low-energy heterotic string effective action [41,25], after truncation (so they can be considered solutions of pure  $N = 4, d = 4$ ) seem to give the same metric in the supersymmetric limit, breaking only 1/2 of the supersymmetries. The exception is the supersymmetric limit of the general rotating solution in Ref. [42] (see also Ref. [43]), but the situation is unclear because the metric was not explicitly written down in the supersymmetric limit.

The new rotating solutions in this class are, therefore, those with  $m^2 - |\mathcal{T}|^2 \neq 0$ .

First of all, we see that, for  $\mathcal{T} = 0$  (that is, constant scalar  $\lambda$ ) one recovers the Kerr–Newman metric with  $m = |q|$ . This was expected since, as we pointed out in Section 2, the usual IWP metrics (embedded in  $N = 4$  supergravity) are obtained when  $\mathcal{H}_1 = i\mathcal{H}_2 = \frac{1}{\sqrt{2}}V^{-1}$ . This metric has a naked ring singularity<sup>18</sup> at  $r = \cos \theta = 0$ . On the other hand, it has  $m^2 - |\mathcal{T}|^2 \neq 0$ , which, according to our central charge analysis at the beginning of this section, should mean that it has only 1/4 of the supersymmetries unbroken when embedded in  $N = 4$  supergravity. We will give a direct proof of this in the next section.

When  $\mathcal{T} \neq 0$  the situation becomes even worse: there are two naked singularities at

$$\hat{\rho} = |\mathcal{T}| \pm \sqrt{|\mathcal{T}|^2 - \alpha^2 \cos^2 \theta}. \quad (67)$$

<sup>18</sup> The fact that it is a ring, and not just a point can be seen by a further shift of the radial coordinate.

These singularities become one ring-shaped singularity ( $\hat{\rho} = 0, \theta = \pi/2$ ) when  $T = 0$ , but for general  $T$ , the range of values of  $\theta$  and  $\hat{\rho}$  such that  $g_{tt}$  and  $g_{\varphi\varphi}$  diverge is bigger and the singularities are in open surfaces satisfying the above equation. The surfaces are closed when  $\theta$  can take all values from 0 to  $\pi$ , that is, when  $|T| \geq |\alpha|$ .

All rotating, supersymmetric, point-like objects in this class of metrics seem to have naked singularities. A similar result was recently obtained in the framework of the low-energy heterotic string effective action in Ref. [25] and previously in a more restricted case in Refs. [43,36] where the difference with the situation in higher dimensions was also discussed.

#### 4. Supersymmetry

In this section we study the unbroken supersymmetries of the SWIP solutions. We first consider only two vector fields. Pure  $N = 4, d = 4$  supergravity [44] has six vector fields that in the supersymmetry transformation laws appear in two different fashions: three of them are associated to the three metrics  $\alpha_{IJ}^a$  and the other three are associated to the three matrices  $\beta_{IJ}^a$  given in Ref. [45]. Our choice [3] is to identify the first vector field with the vector field that couples to  $\alpha_{IJ}^3 \equiv \alpha_{IJ}$  and the second vector field with the one that couples to  $\beta_{IJ}^3 \equiv \beta_{IJ}$ . The corresponding supersymmetry rules are

$$\begin{aligned} \frac{1}{2} \delta \Psi_{\mu I} &= \partial_{\mu} \epsilon_I - \frac{1}{2} \omega_{\mu}^{+ab} \sigma_{ab} \epsilon_I - \frac{i}{4} e^{2\phi} (\partial_{\mu} a) \epsilon_I \\ &\quad - \frac{1}{2\sqrt{2}} e^{-\phi} \sigma^{ab} \left( F_{ab}^{+(1)} \alpha_{IJ} + i F_{ab}^{+(2)} \beta_{IJ} \right) \gamma_{\mu} \epsilon^J, \\ \frac{1}{2} \delta A_I &= -\frac{i}{2} e^{2\phi} \gamma^{\mu} (\partial_{\mu} a) \epsilon_I + \frac{1}{\sqrt{2}} e^{-\phi} \sigma^{ab} \left( F_{ab}^{-(1)} \alpha_{IJ} + i F_{ab}^{-(2)} \beta_{IJ} \right) \epsilon^J. \end{aligned} \quad (68)$$

Making the obvious choice of vierbein one-forms and vectors basis

$$\begin{cases} e^0 = e^{\phi} (dt + \omega_{\underline{i}} dx^{\underline{i}}), \\ e^i = e^{-\phi} dx^{\underline{i}}, \end{cases} \quad \begin{cases} e_0 = e^{-\phi} \partial_{\underline{0}}, \\ e_i = e^{\phi} (-\omega_{\underline{i}} \partial_{\underline{0}} + \partial_{\underline{i}}), \end{cases} \quad (69)$$

the components of the spin-connection one-form are given by

$$\begin{aligned} \omega^{+0i} &= \frac{i}{4} e^{3U} \left[ \partial_{\underline{i}} \bar{V} e^0 + i \epsilon_{ijk} \partial_{\underline{j}} \bar{V} e^k \right], \\ \omega^{+ij} &= \frac{1}{4} e^{3U} \left[ \epsilon_{ijk} \partial_{\underline{k}} \bar{V} e^0 + 2i (\partial_{\underline{i}} \bar{V}) \delta_{\underline{j}\underline{k}} e^k \right], \end{aligned} \quad (70)$$

where  $V = b + ie^{-2U}$ . The curvatures for the vector fields are given by

$$\begin{aligned} F_{0i}^{+(n)} &= -\frac{i}{2} e^{2\phi} B_{\underline{i}}^{(n)}, \\ F_{ij}^{+(n)} &= \frac{1}{2} \epsilon_{ijk} e^{2\phi} B_{\underline{k}}^{(n)}, \end{aligned} \quad (71)$$

with

$$B_{\underline{i}}^{(n)} = \frac{e^{2U}}{\overline{\mathcal{H}}_2} \left[ \partial_{\underline{i}} \overline{\mathcal{H}}_2 (k^{(n)} \mathcal{H}_1 + \overline{k}^{(n)} \overline{\mathcal{H}}_1) - \partial_{\underline{i}} \overline{\mathcal{H}}_1 (k^{(n)} \mathcal{H}_2 + \overline{k}^{(n)} \overline{\mathcal{H}}_2) \right]. \quad (72)$$

We first consider the supersymmetry variation of the gravitino. The variation of the time component leads to (assuming that the Killing spinor is time independent)

$$\begin{aligned} \frac{1}{2} \delta \Psi_{0I} &= -\frac{1}{2} e^U \omega_0^{+ab} \sigma_{ab} \epsilon_I - \frac{1}{4} \sqrt{2} e^{U-\phi} \sigma^{ab} \left( F_{ab}^{+(1)} \alpha_{IJ} + i F_{ab}^{+(2)} \beta_{IJ} \right) \gamma^0 \epsilon^J \\ &= 0, \end{aligned} \quad (73)$$

or

$$\sqrt{2} \left( B_{\underline{i}}^{(1)} \alpha_{IJ} + i B_{\underline{i}}^{(2)} \beta_{IJ} \right) \epsilon^J = e^{-\phi+3U} (\partial_{\underline{i}} \overline{V}) \gamma^0 \epsilon_I. \quad (74)$$

On the other hand, the variation of the space components leads to

$$\begin{aligned} \frac{1}{2} (\delta \Psi_{\underline{i}} - \omega_{\underline{i}} \delta \Psi_0) &= \partial_{\underline{i}} \epsilon_I - \frac{i}{4} e^{2\phi} (\partial_{\underline{i}} a) \epsilon_I + \gamma^j \left[ -e^{-U} \omega_i^{+0j} \gamma^0 \epsilon_I \right. \\ &\quad \left. + \frac{1}{2} \sqrt{2} e^{-U-\phi} (F_{0j}^{+(1)} \alpha_{IJ} + i F_{0j}^{+(2)} \beta_{IJ}) \gamma^i \gamma^0 \epsilon^J \right] \\ &= 0. \end{aligned} \quad (75)$$

Applying the identity

$$\gamma^j \gamma^i F_{0j}^{+(n)} = F_{0i}^{+(n)} - \gamma^j F_{ij}^{+(n)} \gamma^0 \gamma_5, \quad (76)$$

and using the explicit form of  $\omega$  and  $F$ , the requirement that

$$\frac{1}{2} (\delta \Psi_{\underline{i}} - \omega_{\underline{i}} \delta \Psi_0) = 0 \quad (77)$$

leads to the following equation for the Killing spinor:

$$\partial_{\underline{i}} \epsilon_I - \frac{i}{4} e^{2\phi} (\partial_{\underline{i}} a) \epsilon_I + \frac{1}{2} \sqrt{2} e^{-U-\phi} \left( F_{0i}^{+(1)} \alpha_{IJ} + i F_{0i}^{+(2)} \beta_{IJ} \right) \gamma^0 \epsilon^J = 0. \quad (78)$$

Substituting the expression (71) and applying (74) this Killing spinor equation can be rewritten as follows:

$$\partial_{\underline{i}} \epsilon_I - \frac{i}{4} e^{2\phi} (\partial_{\underline{i}} a) \epsilon_I + \frac{i}{4} e^{2U} (\partial_{\underline{i}} \overline{V}) \epsilon_I = 0. \quad (79)$$

Next, we apply the identity

$$e^{2\phi} (\partial_{\underline{i}} a) - e^{2U} (\partial_{\underline{i}} \overline{V}) = \frac{i}{\overline{\mathcal{H}}_2} \partial_{\underline{i}} \overline{\mathcal{H}}_2 - \frac{i}{\mathcal{H}_2} \partial_{\underline{i}} \mathcal{H}_2, \quad (80)$$

and find that

$$\partial_{\bar{i}}\epsilon_I + \frac{1}{4}\left(\frac{\partial_{\bar{i}}\bar{\mathcal{H}}_2}{\bar{\mathcal{H}}_2} - \frac{\partial_{\bar{i}}\mathcal{H}_2}{\mathcal{H}_2}\right)\epsilon_I - \frac{1}{2}(\partial_{\bar{i}}U)\epsilon_I = 0. \quad (81)$$

Finally, we can solve this equation for the Killing spinor as follows:

$$\epsilon_I = \left(\frac{\bar{\mathcal{H}}_2}{\mathcal{H}_2}\right)^{-1/4} e^{U/2} \epsilon_{I(0)}, \quad (82)$$

for constant  $\epsilon_{I(0)}$ . We note that the first factor in the expression for  $\epsilon_I$  containing the harmonic function is exactly what one would expect for an  $SL(2, \mathbb{R})$  covariant spinor [29].

The constant spinors  $\epsilon_{I(0)}$  satisfy certain algebraic conditions which are determined by the vanishing of the supersymmetry rules of the time component of the gravitino (see Eq. (74)) and of  $A_I$ . For clarity, we repeat here Eq. (74), and give the equation that follows from the vanishing of  $\delta A_I$ :

$$\begin{aligned} \sqrt{2}\left(B_i^{(1)}\alpha_{IJ} + iB_i^{(2)}\beta_{IJ}\right)\epsilon^J &= e^{-\phi+3U}(\partial_{\bar{i}}\bar{V})\gamma^0\epsilon_I, \\ \sqrt{2}\left(\bar{B}_i^{(1)}\alpha_{IJ} + i\bar{B}_i^{(2)}\beta_{IJ}\right)\epsilon^J &= e^{\phi+U}(\partial_{\bar{i}}\lambda)\gamma^0\epsilon_I. \end{aligned} \quad (83)$$

In terms of the harmonic functions these two equations read as follows:<sup>19</sup>

$$\begin{aligned} [k^{(1)}\mathcal{A}_{\bar{i}} + \bar{k}^{(1)}\bar{\mathcal{B}}_{\bar{i}}]\alpha_{IJ}\epsilon_{(0)}^J + i[k^{(2)}\mathcal{A}_{\bar{i}} + \bar{k}^{(2)}\bar{\mathcal{B}}_{\bar{i}}]\beta_{IJ}\epsilon_{(0)}^J &= -\mathcal{A}_{\bar{i}}\gamma^0\epsilon_{I(0)}, \\ [\bar{k}^{(1)}\bar{\mathcal{A}}_{\bar{i}} + k^{(1)}\mathcal{B}_{\bar{i}}]\alpha_{IJ}\epsilon_{(0)}^J + i[\bar{k}^{(2)}\bar{\mathcal{A}}_{\bar{i}} + k^{(2)}\mathcal{B}_{\bar{i}}]\beta_{IJ}\epsilon_{(0)}^J &= -\bar{\mathcal{B}}_{\bar{i}}\gamma^0\epsilon_{I(0)}, \end{aligned} \quad (84)$$

with

$$\begin{aligned} \mathcal{A}_{\bar{i}} &= \mathcal{H}_1\partial_{\bar{i}}\bar{\mathcal{H}}_2 - \mathcal{H}_2\partial_{\bar{i}}\bar{\mathcal{H}}_1, \\ \mathcal{B}_{\bar{i}} &= \mathcal{H}_1\partial_{\bar{i}}\mathcal{H}_2 - \mathcal{H}_2\partial_{\bar{i}}\mathcal{H}_1. \end{aligned} \quad (85)$$

Let us now define

$$\mathcal{C}_{\bar{i}} \equiv \mathcal{A}_{\bar{i}} - e^{2i\delta}\bar{\mathcal{B}}_{\bar{i}}, \quad (86)$$

for some real and non-yet specified parameter  $\delta$ . Next, we solve the two constrained complex parameters  $k^{(1)}$  and  $k^{(2)}$  in terms of a single real parameter  $\gamma$  as follows:

$$k^{(1)} = \frac{1}{2}e^{i\gamma}, \quad k^{(2)} = \frac{i}{2}e^{i\gamma}. \quad (87)$$

Using this parametrization of  $k^{(1)}$  and  $k^{(2)}$ , it is easy to see that Eqs. (84) reduce to

<sup>19</sup> To derive this equation we must make a choice of convention for the branch cuts of the square root of a complex number. In the specific calculation below we have made use of the identity  $\bar{\mathcal{H}}_2/\mathcal{H}_2^{1/2} = \bar{\mathcal{H}}_2^{1/2}$ . The effect of taking another sign at the r.h.s. of this equation is that in the final answer for the Killing spinors  $\gamma^0 \rightarrow -\gamma^0$ .

$$\mathcal{A}_i \xi_I - \mathcal{C}_i \eta_I = 0, \quad (88)$$

$$\overline{\mathcal{A}}_i \zeta_I - \overline{\mathcal{C}}_i \eta_I = 0, \quad (89)$$

where  $\xi_I$ ,  $\eta_I$  and  $\zeta_I$  are constant spinors defined as

$$\eta_I = \frac{1}{2} e^{-i(\gamma+\delta)} (\alpha_{IJ} + \beta_{IJ}) \epsilon_{(0)}^J, \quad (90)$$

$$\zeta_I = \frac{1}{2} e^{i(\gamma+\delta)} (\alpha_{IJ} - \beta_{IJ}) \epsilon_{(0)}^J + e^{i\delta} \gamma^0 \epsilon_{I(0)}, \quad (91)$$

$$\xi_I = [\cos(\gamma + \delta) \alpha_{IJ} - i \sin(\gamma + \delta) \beta_{IJ}] \epsilon_{(0)}^J + e^{i\delta} \gamma^0 \epsilon_{I(0)} \quad (92)$$

$$= \eta_I + \zeta_I. \quad (93)$$

We have to consider several different cases:

- (i) If both  $\mathcal{A}_i$  and  $\mathcal{C}_i$  are different from zero for any possible value of  $\delta$  and if they are also different for any value of  $\delta$ , then, since the spinors  $\xi_I, \eta_I, \zeta_I$  are constant, they have to vanish. Given that  $\xi_I = \eta_I + \zeta_I$ , this gives only two independent conditions:  $\eta_I = \zeta_I = 0$  on the constant spinors  $\epsilon_{I(0)}$ .

By making use of the explicit form of the matrices  $\alpha_{IJ}$  and  $\beta_{IJ}$  (see Ref. [3]), we can solve these two equations and we find that the constant parts of the Killing spinors are

$$\epsilon_{(0)}^3 = \epsilon_{(0)}^4 = 0, \quad \epsilon_{1(0)} = e^{i\gamma} \gamma^0 \epsilon_{(0)}^2. \quad (94)$$

This is the generic case and we just have proven that only 1/4 of the supersymmetries are unbroken in this case.

- (ii) If both  $\mathcal{A}_i$  and  $\mathcal{C}_i$  are equal to each other and different from zero for some value of  $\delta$ , then, using that  $\xi_I = \eta_I + \zeta_I$  we get again the same equations and the same amount of unbroken supersymmetry.
- (iii) If  $\mathcal{A}_i$  vanishes we are again in the same case.
- (iv) If  $\mathcal{C}_i$  vanishes for some value of  $\delta$ , then we only get one condition on the constant spinors  $\epsilon_{I(0)}$ :  $\xi = 0$ . In this case, and only in this case, 1/2 of the supersymmetries are unbroken.

This proves that we always have that *at least 1/4 of the supersymmetries are unbroken and all the SWIP metrics admit Killing spinors.*

We can now study how different particular cases fit into this scheme. We can treat first the case of a single point-like object because we know in which cases one or two Bogomol'nyi bounds are saturated. For one bound saturated we expect two algebraic constraints on the Killing spinors and for two bounds saturated we expect only one (1/4 and 1/2 of the supersymmetries unbroken respectively). Substituting Eqs. (30) and (32) of  $\mathcal{H}_1$  and  $\mathcal{H}_2$  in Eqs. (30) and (32), which are expressed in terms of the charges, we find that

$$\begin{aligned} \mathcal{A}_i &= i \left\{ \overline{\mathcal{M}} \partial_i \frac{1}{r} + (|\mathcal{M}|^2 - |\mathcal{T}|^2) \frac{1}{r} \partial_i \frac{1}{r} \right\}, \\ \mathcal{B}_i &= i \mathcal{T} \partial_i \frac{1}{r}. \end{aligned} \quad (95)$$

It is easy to see that when  $|\mathcal{M}|^2 - |\mathcal{T}|^2 = 0$ , then  $\mathcal{M} = e^{i2\delta}\overline{\mathcal{T}}$  and we are in the case in which  $\mathcal{C}_i = 0$  and  $1/2$  of the supersymmetries are unbroken.  $|\mathcal{M}|^2 - |\mathcal{T}|^2 \neq 0$  is the generic case and only  $1/4$  of the supersymmetries are unbroken, as expected from the Bogomol'nyi bounds analysis.

Another particular case is when  $\mathcal{H}_1 = e^{i\sigma}\mathcal{H}_2$  for some real constant  $\sigma$ . Then  $\mathcal{B}_i = 0$  and  $\mathcal{C}_i = \mathcal{A}_i$  and only  $1/4$  of the supersymmetries are unbroken. In the  $N = 2, d = 4$  IWP solutions

$$\mathcal{H}_1 = i\mathcal{H}_2 = \frac{1}{\sqrt{2}}V^{-1}, \quad (96)$$

that is  $\sigma = \pi/2$ , and, as expected, they have only  $1/4$  of the supersymmetries unbroken when embedded in  $N = 4, d = 4$  supergravity.

## 5. Conclusion

We have presented a family of supersymmetric stationary solutions that generalizes to  $N = 4$  supergravity the IWP solutions of  $N = 2$  supergravity in the sense that they are the most general solutions of its kind and that they are manifestly invariant (as a family) under all the duality symmetries of  $N = 4, d = 4$  supergravity.

We have studied the supersymmetry properties of the general solution and the geometry of the most general point-like solution in this class, arriving to the conclusion that no rotating supersymmetric black holes exist in pure  $N = 4, d = 4$  supergravity (see also Refs. [8,43,25]). The situation in four dimensions seems to be radically different from the situation in five dimensions, where supersymmetric rotating black holes with regular horizon have been found in Ref. [36]. The physical reason for this has not yet been understood.

We have also argued that the interpretation of our solution for the two-vectors case as a solution of  $N = 2, d = 4$  supergravity coupled to one vector field and in terms of special geometry provides a most interesting clue to get the most general supersymmetric stationary solution of  $N = 2, d = 4$  supergravity coupled to any number of vector multiplets.

We have not studied other solutions in this class describing extended objects like strings and membranes. These can be obtained by choosing complex harmonic functions that depend on less than three spatial coordinates. These solutions are, of course, also supersymmetric because the general analysis performed in Section 4 applies to them. They are also manifestly  $S$  and  $T$  duality invariant. We have not discussed massless solutions either. Exploiting this class of solutions to its full extent will require much more work, but all the results that will eventually be obtained will also be manifestly duality invariant. We believe that this is really progress towards a full characterization of the most general supersymmetric solution in  $N = 8$  supergravity, which should be manifestly  $U$  duality invariant. Having this solution at hand would be of immense

value as a testing ground for all the current ideas on duality and the construction of supersymmetric black holes from  $D$  branes.

It would also be interesting to take these solutions out of the supersymmetric limit. The non-supersymmetric metrics should also be manifestly duality invariant and would describe single stationary black holes, asymptotically Taub-NUT metrics, strings or membranes because there would be no balance of forces and it would be impossible to have more than one of such objects in equilibrium. Many of these non-supersymmetric solutions are known in the point-like case (black holes or asymptotically Taub-NUT metrics) (see, for instance, Ref. [5]). This is, we hope, one of the main goals of obtaining supersymmetric solutions. They are easier to obtain because of the additional (super) symmetry that constrains the equations of motion and they should help in finding the general non-supersymmetric solutions which are, perhaps, the most interesting from a physical point of view.

After the completion of this work we discovered that some of the stationary solutions presented in this work have also been found, in a different setting, in Ref. [46], by directly solving the Killing spinor equations.

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